Stochastic resonance without an external periodic drive in a simple prey-predator model

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We have investigated the effect of noise on a simple prey-predator model where the oscillations are triggered by the internal dynamics of the system without the aid of any external periodic drive. We report the occurrence of stochastic resonancelike behavior in this system, which does not have a threshold or a potential barrier, in the absence of an external drive.

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I. INTRODUCTION

Since the introduction of the phenomenon of stochastic resonance (SR) [1] to explain the periodicity of ice ages, it has been studied extensively [2,3] and used to explain a wide variety of phenomena [4]. In most studies it has been shown that the presence of noise in a bistable system driven by a subthreshold external field allows coherent interwell transitions and thus strongly amplifies the signal. Apart from bistable systems it has been observed in monostable systems [5], random walk [6], excitable systems [7], discrete time maps [8], neural network based models [9], etc. But in all these studies the system is driven by a subthreshold external drive.

In a numerical study, Gang *et al.* [10] have shown SR to occur even in the absence of an external periodic drive. They have studied noise induced effects in a two-dimensional autonomous system in which a limit cycle appears as some parameter is varied. In the present paper we show that SR-like behavior is exhibited by an autocatalytic reaction model whose internal dynamics alone can trigger an oscillation even when no other periodic drive is present. A very important feature of the system under investigation here is that it does not have a threshold and the process does not describe hopping through a potential barrier. Recently several dynamical or nondynamical systems without threshold have been reported to utilize noise to amplify a weak periodic input signal [11,12]. In Sec. II we define our model. The numerical simulations and their results are presented in Sec. III. The results are discussed in Sec. IV followed by conclusions in Sec. V.

II. PREY-PREDATOR MODEL

Oscillating or periodic phenomenon are ubiquitous in physics, chemistry, astronomy, and biology. One of the widely studied and simplest model that mimics a variety of oscillating processes is the prey-predator model [13]. It has a simple mechanism and most of the oscillating chemical models developed later [14] are modifications of this model by Lotka [13],

$$A+X\xrightarrow{k_1}2X, \tag{1}$$

$$X+Y\xrightarrow{k_2}2Y,$$

$$Y\xrightarrow{k_3}B.$$

The concentrations of the initial reactant A and the final product B are maintained constant in time.

The deterministic equations for this system are

$$\frac{dx}{dt}=k_1ax-k_2xy,$$

$$\frac{dy}{dt}=k_2xy-k_3y, \tag{2}$$

where a, x, and y are the concentrations of species A, X, and Y, respectively. The stationary state for this system is $x_s=k_3/k_2$ and $y_s=k_1a/k_2$.

The stability analysis [15] of this steady state gives a periodic motion for any small deviation from the steady state, i.e., the system is marginally stable. The concentration of X and Y oscillate around the steady state with certain frequency that is independent of the initial conditions. We call it the characteristic frequency of the system (Ω). In the $x-y$ phase space, there are an infinite number of periodic orbits around the steady state.

We can convert the system to a Hamiltonian system with a logarithmic change of variables and define a constant of motion, say energy E, of the system as [16]:

$$E=k_2y+k_3x-k_1a\log_e y-k_3\log_e x. \tag{3}$$

The orbits that are closer to steady state have a lower energy. For a deterministic system, the value of E remains constant in time.

Realistic oscillatory systems do not follow the solutions of the above differential equations exactly. The concentrations of various species fluctuate around their deterministic values due to environmental noise. Noise may be due to external factors like fluctuations in incoming flux or temperature, or internal (due to finite system size [17], etc.). The simplest way to include these fluctuations is to add a noise term to the above deterministic equation and the equivalent stochastic differential equation will be,

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$$\frac{dx}{dt}=k_1ax-k_2xy+\eta_1(t),$$

$$\frac{dy}{dt}=k_2xy-k_3y+\eta_2(t), \tag{4}$$

where $\eta_i(t)$ is a uniform white noise with $\langle \eta_i(t)\rangle=0$ and $\langle \eta_i(t)\eta_j(t')\rangle=D\delta_{ij}\delta(t-t')$; $i,j=1,2$.

It is not possible to deal with the above coupled differential equations analytically, so we will treat them numerically.

III. NUMERICAL SIMULATION OF A NOISY PREY-PREDATOR MODEL

We have performed numerical simulations of the system (3) with $k_1=1$, $k_2=2$, and $k_3=2$. The concentration of A is $a=2$ and is maintained constant externally. The differential equations were solved using fourth-order Runge Kutta method [18] and an average of over 1000 realizations were taken. Though we start with the same initial conditions for all these trajectories we record the time series only after 2000 steps (discarding them as transients) and at this time all the trajectories will have different values of x and y (though they may be close). We record the average concentration of x for 8192 time steps. In the presence of noise, the axes of $x-y$ phase space may be crossed, i.e., the concentrations may become negative. One way to overcome this problem of negative concentrations is to use multiplicative noise [19]. In an autonomous system, for low noise intensities, multiplicative and additive noise [20] yield similar results. In our simulations we use additive noise and discard those values of noise for which either x or y becomes negative. Under these conditions, it is observed that increasing the number of time steps does not alter the results significantly. The reason may be that if the system is very near the steady state, a small noise does not drive it to a very far off value. However, a very strong noise may take it to a far from equilibrium state. There is a possibility of getting a different result when one of the concentrations goes to zero within numerical precision (as a result of this the other concentration may either approach zero or infinity).

If the system is initially present close to the steady state, the concentrations of X and Y show oscillations in time and the phase space shows eliptical orbits around the steady state. The effect of noise is to take the system to the neighboring deterministic orbits (Fig. 1). As a result the energy of the system no longer remains constant. In Fig. 2, we show the energy as a function of time for the noisy prey-predator system. In the presence of weak noise the energy shows negligible fluctuations. The value for the deterministic system when it is at steady state is 4.0. When the noise intensity is high the energy of the system shows comparatively large fluctuations (Fig. 2). The noise sometimes takes the system away from the steady state and sometimes it brings it back near the steady state.

To obtain the various frequencies contributing to the dynamics of the system, we perform the power spectrum analysis. For the system that is initially present at the steady state, i.e., $x_i=1.0$ and $y_i=1.0$, the power spectrum does not show discretely singular contributions from any frequency. But due to noise present in the environment it is bound to move away from the steady state and the concentrations, x and y, start showing oscillatory behavior. Therefore, it makes more sense to study the system when it is initially present close to steady state. In that case, the power spectrum shows a peak at the characteristic frequency Ω of the system. For very low noise, the spectrum peak is low (Fig. 3). But as we increase the noise intensity D, the height of the peak increases. This is a significant observation indicating the possibility of the occurrence of SR. Also, with increase in noise intensity the higher harmonics of the fundamental frequency appear (Fig. 3) and their heights also increase with an increase in D. If we still increase D, the power spectrum shows a broad peak with large background noise (Fig. 3).

Figure 4 shows a plot of signal-to-noise ratio (SNR) vs noise intensity D. The SNR has been defined as the ratio of

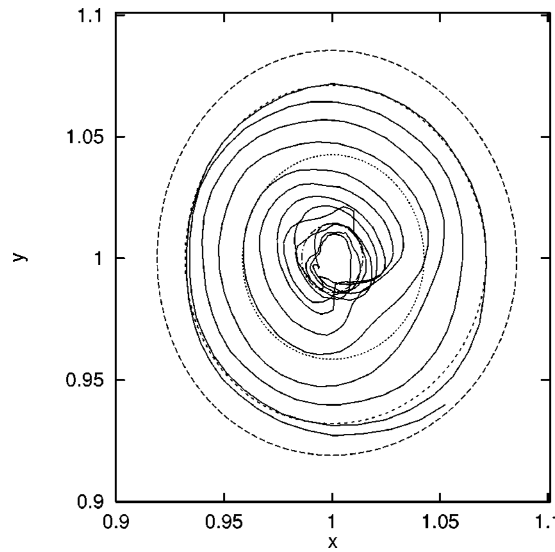

FIG. 1. Trajectories for different initial conditions with noise (solid line) and without noise (dots and dashes, dots, small dashes, big dashes). The noisy trajectory starting from the initial condition $x_i=1.001$; $y_i=1.001$ visits various deterministic orbits. Values of the other parameters are $a=2$, $k_1=1$, $k_2=2$, and $k_3=2$.

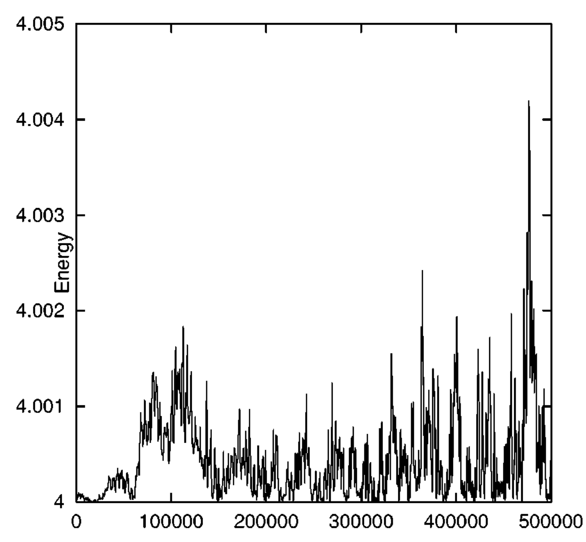

FIG. 2. Constant of motion E vs time for $x_i=1.001$, $y_i=1.001$, $a=2$, $k_1=1$, $k_2=2$, $k_3=2$, and $D=4.0$.

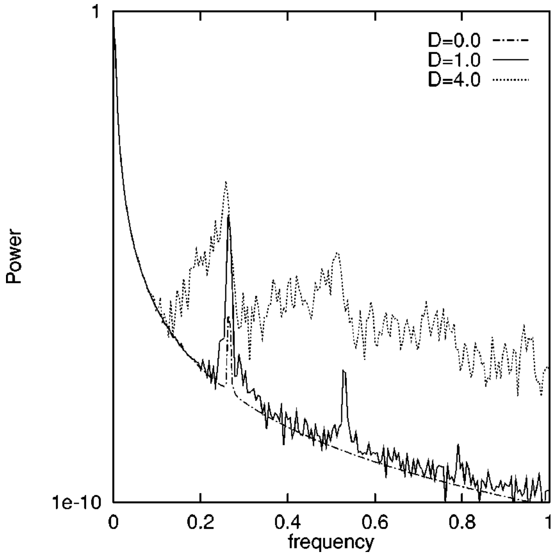

FIG. 3. Power spectra of the above system initially present at $x_i = 1.001$, and $y_i = 1.001$ for different D ($a = 2$, $k_1 = 1$, $k_2 = 2$, and $k_3 = 2$).

intensity of the peak in the power spectrum to the height of the noisy background $Q(\Omega)$ at the same frequency.

$$\mathrm{SNR} = \log_{10}\left[\frac{\text{total power in the frequency } \Omega}{Q(\Omega)}\right]. \tag{5}$$

Similarly, the SNR of the second harmonic is defined. We observe maxima in SNR of the first and second harmonics plotted against the noise intensity D. This indicates that the noise has induced coherent motion in the system that was initially at the steady state and this coherence is a maximum for an optimum noise intensity.

These results clearly demonstrate the phenomenon of SR, which has been stimulated by noise alone in a system whose deterministic dynamics are autonomous.

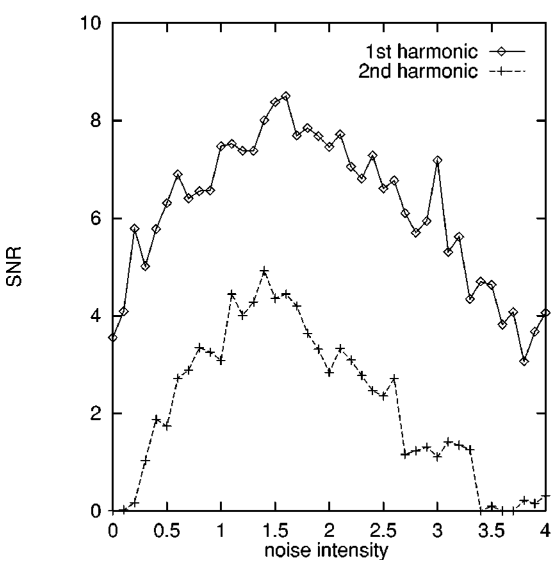

FIG. 4. SNR vs D for the first and second harmonics for the system initially at $x_i = 1.001$; $y_i = 1.001$ (all the parameters are the same as in Fig. 2).

IV. DISCUSSION OF RESULTS

SR in an autonomous system has been demonstrated in numerical studies on a 2D system, [10] which exhibits a stable limit cycle for certain values of a control parameter. In this regime, the power spectrum shows a δ function at the frequency of the limit cycle. Noise was found to broaden the peak and shift it towards higher frequency. In another regime of the control parameter, the system has four fixed points, two stable and two unstable. Noise is found to induce a peak in its power spectrum at a definite frequency and the peak shifts to a higher frequency with increase in noise intensity. Explanations for these results were given by Rappel and Strogatz [21]. However, in our system, noise does not effect the position of the peak in the power spectrum. The power spectrum shows the peak at the same frequency for any value of noise strength.

The crucial property of the system studied by Gang *et al.* [10] is the infinite-period bifurcation mechanism for the formation of the limit cycle. Due to the saddle node bifurcation, the motion along the limit cycle is highly nonuniform. It was suggested [21], that this kind of SR will not occur in a system whose bifurcations are created by a Hopf bifurcation.

Apart from this, there is another very important difference between our model and the other systems. Most of the systems that have been explored for the occurrence of SR, including that of Gang *et al.* [10], have a common feature: They have three fixed points, an unstable point between two stable points. Due to this, these models have a threshold or a potential barrier and the processes describe the hopping through a potential barrier. However, in the past few years the role of noise and periodic drive in nondynamical systems has been investigated. These studies show that SR can occur even in threshold free systems [11], i.e., those systems that are able to respond to input signals of arbitrarily small amplitude. In contrast, in a system with a threshold, a subthreshold forcing does not generate any output. These studies show that the addition of external noise to a periodic input signal may result in enhancement of the response of the system. But no external drive is needed in the Lotka-Volterra model studied here because the internal dynamics of the system generates a periodic signal. The prey-predator model has marginally stable periodic solutions for small perturbations away from steady state. The effect of noise is to trigger oscillations between the neighboring orbits with a time scale that corresponds to a characteristic frequency. It is interesting to see that an increase in the noise intensity induces more coherent motion resulting in a sharper peak in the power spectrum.

V. CONCLUSION

In the present paper we have studied the behavior of a simple model of oscillating chemical system in the presence of noise, which may be external or internal. Although, a noisy prey-predator system will ultimately go to the state where either both the concentrations are zero or where y is zero and x escapes to infinity, it is interesting to observe the occurrence of SR-like behavior in such a system in the short-time limit (even in the absence of an external periodic drive). In an interesting work by Lipowski [22], SR without an external drive has been reported in a lattice prey-predator model using Monte-Carlo simulations. In an investigation on interplay between noise and periodic modulations in the Lotka-Volterra model of two species competition, Vilar and Sole [19] have demonstrated that the presence of noise is responsible for the generation of temporal oscillations and for the appearance of spatial patterns that do not arise in the deterministic model. This was the first example of SR in a model of population dynamics. In their paper, a control parameter, which accounts for the interactions among the species, is periodically modulated and noise helps the system to respond to this modulation. However in the paper reported here the system has its characteristic time scale and does not require any external drive.

These studies show that, contrary to our belief, environmental noise does not always play a destructive role of wiping out the coherent behavior in such systems. There are a wide range of chemical and biological systems that can be modeled by the above system and SR without external periodic drive may have interesting applications there.

We would like to emphasize that the occurrence of SR in the above system indicates that it is a generic phenomenon and the presence of an external drive or a system having a threshold or potential barrier is not essential for its occurrence. It will be interesting to study the effect of realistic noise on these systems. In many situations the time scale of noise is comparable to that of the system where it becomes important to investigate the effect of noise correlation. We are extending this paper to other systems with stable limit cycles or more complex features as solutions.

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